A VAN DER CORPUT LEMMA FOR THE p-ADIC NUMBERS

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Abstract. We prove a version of van der Corput's Lemma for polynomials over the p-adic numbers.

1. Introduction

The following lemma goes back to J.G. van der Corput in [3]. It has many applications in number theory and harmonic analysis. In particular, it is key to the study of oscillatory integrals (see [6]). We note that only partial van der Corput type lemmas are known in dimensions greater than one (see [2]). As a consequence the theory of oscillatory integrals in higher dimensions is relatively open.

Lemma 1. *Suppose that $f:(a,b)\to\mathbb{R}$ is n times differentiable, where $n\geq 2$, and $|f^{(n)}(x)|\geq\lambda>0$ on (a,b). Then*

$$\left|\int_a^b e^{if(x)}\,dx\right|\leq C_n\frac{n}{\lambda^{1/n}},$$

where $C_n\leq 2^{5/3}$ for all $n\geq 2$ and $C_n\to 4/e$ as $n\to\infty$.

It can be shown by considering $f(x)=x^n$ that the linear growth in n is optimal, and this more precise formulation is due to G.I. Arhipov, A.A. Karacuba and V.N. Cubarikov [1]. Consideration of the Chebyshev polynomials shows that the constant becomes sharp as n tends to infinity (see [5]). The following corollary can be easily obtained using Stirling's formula.

Corollary 2. *Let $f(x)=a_0+a_1x+\cdots+a_nx^n$ be a real polynomial of degree $n\geq 1$. Then*

$$\left|\int_a^b e^{if(x)}\,dx\right|\leq\frac{2^{5/3}e}{|a_n|^{1/n}}<\frac{9}{|a_n|^{1/n}}$$

for all $a,b\in\mathbb{R}$.

We will prove a p-adic version of this corollary, opening the way for the study of oscillatory integrals on the p-adics. This problem was first considered by J. Wright [8], where lemmas for polynomials of degree two and monomials of degree three were proven.

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2. Introduction to the p-Adic Numbers

For a more complete introduction to the p-adic numbers, see [4] or [7]. Here we will outline what we will need.

Fix a prime number p. Any non-zero rational number x can be uniquely expressed in the form $p^k m/n$, where m and n have no common divisors and neither is divisible by p. We then define the p-adic norm on the rational numbers by $|x| = p^{-k}$ when $x \neq 0$, and $|0| = 0$. We obtain the p-adic numbers by completing $\mathbb{Q}$ with respect to this norm. It is not difficult to show that the norm satisfies the following properties:

$$|xy| = |x||y|$$
$$|x + y| \leq \max\{|x|, |y|\}.$$

It follows from the second property that

$$\{y : |y - x_1| \leq p^r\} = \{y : |y - x_2| \leq p^r\}$$

when $|x_1 - x_2| \leq p^r$, so every point within a ball can be considered to be its centre.

A nonzero p-adic number x with $|x| = p^{-k}$, may be written in the form

$$x = \sum_{j=k}^{\infty} x_j p^j,$$

where $0 \leq x_j \leq p-1$, and $x_k \neq 0$. This will be called the standard p-adic expansion and the arithmetic is done formally with carrying. Define $\chi : \mathbb{Q}_p \to \mathbb{C}$ by

$$\chi(x) = \begin{cases} \prod_{j=k}^{-1} e^{2\pi i x_j / p^j} & |x| > 1 \\ 1 & |x| \leq 1. \end{cases}$$

The characters of $\mathbb{Q}_p$ are all of the form $\chi_\epsilon : \mathbb{Q}_p \to \mathbb{C}$;

$$\chi_\epsilon(x) = \chi(\epsilon x),$$

where $\epsilon \in \mathbb{Q}_p$. Finally $\mathbb{Q}_p$ is a locally compact commutative group, so there is a Haar measure, that necessarily satisfies $d(ax) = |a|dx$. We normalise the measure so that $\{x \in \mathbb{Q}_p : |x| \leq p^r\}$ has measure p^r.

The usual arguments can be employed to obtain the standard Fourier results.

3. p-Adic van der Corput Lemmas

The main thrust of this section is to prove the following lemma for p-adic polynomials. The Euclidean arguments will not be helpful as there is no order on the p-adic numbers.

Lemma 1. *Suppose that $a_0, \ldots, a_n \in \mathbb{Q}_p$. Then*

$$\left| \int_{|x| \leq 1} \chi(a_1 x + \cdots + a_n x^n)\, dx \right| \leq \frac{p^m}{|m a_m|^{1/m}},$$

where $m = \max\{l : |l a_l| \geq |j a_j| \text{ for all } j \neq l\}$.

Before proving Lemma 1 we note some easy corollaries.

Corollary 2. *Suppose that $a_0, \dots, a_n \in \mathbb{Q}_p$. Then*

$$\left| \int_{|x|\le 1} \chi(a_1 x + \cdots + a_n x^n)\, dx \right| \le \frac{2p^n}{\lambda^{1/n}},$$

where $\lambda = \max_{1\le j\le n} |a_j|$.

Proof. Suppose that $|a_k| = \max_{1\le j\le n} |a_j| = \lambda$. By Lemma 1 we have

$$|I| = \left| \int_{|x|\le 1} \chi(a_1 x + \cdots + a_n x^n)\, dx \right| \le \frac{p^m}{|m a_m|^{1/m}},$$

where $m = \max\{l : |l a_l| \ge |j a_j| \quad \text{for all} \quad j \ne l\}$. Now as $|m a_m| \ge |k a_k|$, we have

$$|I| \le \frac{p^m}{|k a_k|^{1/m}} \le \frac{k^{1/m} p^m}{|a_k|^{1/m}} \le \frac{n^{1/m} p^m}{\lambda^{1/n}} \le \frac{2p^n}{\lambda^{1/n}},$$

and we are done. □

From this we obtain our main result which holds uniformly over all balls. It is the p-adic equivalent of Corollary 2 in Section 1.

Corollary 3. *Suppose that $x_0, a_0, \dots, a_n \in \mathbb{Q}_p$ and $r \in \mathbb{Z}$. Then*

$$\left| \int_{|x-x_0|\le p^r} \chi(a_0 + a_1 x + \cdots + a_n x^n)\, dx \right| \le \frac{2p^n}{|a_n|^{1/n}}.$$

Proof. Let $y = p^r(x - x_0)$, so that

$$\begin{aligned}
I &= \int_{|x-x_0|\le p^r} \chi(a_0 + a_1 x + \cdots + a_n x^n)\, dx \\
&= \int_{|y|\le 1} \chi\left(a_0 + a_1\left(\frac{y}{p^r} + x_0\right) + \cdots + a_n\left(\frac{y}{p^r} + x_0\right)^n \right) dx \\
&= p^r \int_{|y|\le 1} \chi\left(b_0(x_0) + \cdots + \frac{b_{n-1}(x_0) y^{n-1}}{p^{(n-1)r}} + \frac{a_n y^n}{p^{nr}} \right) dy \\
&=: p^r I_1,
\end{aligned}$$

where $b_j(x_0) = a_j + \binom{j+1}{j} a_{j+1} x_0 + \cdots + \binom{n}{j} a_n x_0^{n-j}$. We also note that

$$\begin{aligned}
|I_1| &= \left| \chi(b_0(x_0)) \int_{|y|\le 1} \chi\left(\frac{b_1(x_0) y}{p^r} + \cdots + \frac{b_{n-1}(x_0) y^{n-1}}{p^{(n-1)r}} + \frac{a_n y^n}{p^{nr}} \right) dy \right| \\
&= \left| \int_{|y|\le 1} \chi\left(\frac{b_1(x_0) y}{p^r} + \cdots + \frac{b_{n-1}(x_0) y^{n-1}}{p^{(n-1)r}} + \frac{a_n y^n}{p^{nr}} \right) dy \right| \\
&=: |I_2|.
\end{aligned}$$

Thus

$$|I| = p^r |I_2| \le p^r \frac{2p^n}{|p^{-nr} a_n|^{1/n}} = \frac{2p^n}{|a_n|^{1/n}},$$

by Corollary 2. □

We now turn to the proof of Lemma 1. We will need some preliminary lemmas. Our starting point is a consequence of the fact that balls in $\mathbb{Q}_p$ have multiple centres.

Lemma 4. *Suppose that $a \in \mathbb{Q}_p$ and $|a| > 1$. Then*
$$\int_{|x|\leq 1} \chi(ax)\,dx = 0.$$

Proof. First we consider the standard expansion of a, so that
$$a = \sum_{j=-k}^{\infty} a_j p^j,$$
where $k \geq 1$ and $a_{-k} \neq 0$. Now as
$$\{x : |x| \leq 1\} = \{x : |x - p^{k-1}| \leq 1\},$$
we have
$$I = \int_{|x|\leq 1} \chi(ax)\,dx = \int_{|x-p^{k-1}|\leq 1} \chi(ax)\,dx.$$
If we let $y = x - p^{k-1}$, we see that
$$I = \int_{|y|\leq 1} \chi(a(y + p^{k-1}))\,dy = \chi(ap^{k-1}) \int_{|y|\leq 1} \chi(ay)\,dy,$$
so that
$$I = \chi(ap^{k-1})I.$$
Now as $\chi(ap^{k-1}) = e^{2\pi i a_{-k}/p} \neq 1$, we see that $I = 0$. □

If we let $f(y) = a_0 + a_1 y + \cdots + a_n y^n$, then we denote
$$b_j(y) = \frac{f^j(y)}{j!} = a_j + \binom{j+1}{j} a_{j+1} y + \cdots + \binom{n}{j} a_n y^{n-j}. \tag{1}$$
We will use this notation throughout.

Lemma 5. *Suppose that $|ma_m| > |ja_j|$ for all $j > m$, and $|y| \leq 1$. Then*
$$|mb_m(y)| = |ma_m| > |jb_j(y)|$$
for all $j > m$, where b_j is given by (1).

Proof. Suppose that $|ma_m| > |ja_j|$ for all $j > m$. Then
$$|ma_m| > \left|\binom{j-1}{m-1}\right| |ja_j|,$$
so that
$$|a_m| > \left|\binom{j}{m} a_j\right|$$
for all $j > m$. Thus
$$|mb_m(y)| = |m| \left| a_m + \binom{m+1}{m} a_{m+1} y + \cdots + \binom{n}{m} a_n y^{n-m} \right| = |ma_m|$$
for all $|y| \leq 1$. Similarly, if $k > j > m$, then
$$|ma_m| > \left|\binom{k-1}{j-1}\right| |ka_k|,$$
so that
$$|ma_m| > \left| j \binom{k}{j} a_k \right|.$$

Putting these together,

$$|mb_m(y)| = |ma_m| > \left| ja_j + j\binom{j+1}{j} a_{j+1}y + \cdots + j\binom{n}{j} a_n y^{n-j} \right| = |jb_j(y)|$$

for all $|y| \leq 1$. □

Lemma 6. *Suppose that $|a_1| > p$ and $|a_1| > |ja_j|$ for $j > 1$. Then*

$$\int_{|x|\leq 1} \chi(a_1 x + \cdots + a_n x^n)\, dx = 0.$$

Proof. Let $|a_1| = p^{k+1}$ where $k \geq 1$. We split the integral into p^k pieces, so that

$$I = \sum_{y=0}^{p^k-1} \int_{|h|\leq p^{-k}} \chi(a_1(y+h) + \cdots + a_n(y+h)^n)\, dh.$$

Now

$$I = \sum_{y=0}^{p^k-1} \chi(a_1 y + \cdots + a_n y^n) I_1(y),$$

where

$$\begin{aligned} I_1(y) &= \int_{|h|\leq p^{-k}} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_n h^n)\, dh \\ &= \frac{1}{p^k} \int_{|x|\leq 1} \chi(b_1(y)p^k x + \cdots + b_{n-1}(y)p^{(n-1)k} x^{n-1} + a_n p^{nk} x^n)\, dx, \end{aligned}$$

and b_j is given by (1). When $|y| \leq 1$, we have

$$|b_1(y)| = |a_1| > |jb_j(y)|$$

for all $j > 1$, by Lemma 5. Hence

$$|b_1(y)p^k| = \frac{|a_1|}{p^k} = p,$$

and

$$|jb_j(y)p^{jk}|p^{jk} < |b_1(y)p^k|p^k = p^{k+1}.$$

So if $j > 1$, then

$$|b_j(y)p^{jk}| \leq \frac{1}{|j|p^{(j-1)k}} \leq \frac{j}{p^{(j-1)k}} \leq \frac{2}{2^{(2-1)1}} = 1.$$

Thus by Lemma 4,

$$\begin{aligned} I_1(y) &= \frac{1}{p^k} \int_{|x|\leq 1} \chi(b_1(y)p^k x)\chi(b_2(y)p^{2k} + \cdots + a_n p^{nk} x^n)\, dx \\ &= \frac{1}{p^k} \int_{|x|\leq 1} \chi(b_1(y)p^k x)\, dx = 0 \end{aligned}$$

for all $|y| \leq 1$, and we are done. □

Lemma 7. *Suppose that $|ma_m| > p^2$ and $|ma_m| > |ja_j|$ for all $j \neq m$. Then*

$$\int_{|x|\leq 1} \chi(a_1 x + \cdots + a_n x^n)\, dx = \frac{1}{p} \int_{|x|\leq 1} \chi(a_1 p x + \cdots + a_n p^n x^n)\, dx.$$

Proof. We split the integral into p pieces, so that

$$I = \int_{|x|\le 1} \chi(a_1x + \cdots + a_nx^n)\, dx = \sum_{y=0}^{p-1} \chi(a_1y + \cdots + a_ny^n) I_1(y),$$

where

$$\begin{aligned} I_1(y) &= \int_{|h|\le 1/p} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_nh^n)\, dh \\ &= \frac{1}{p}\int_{|x|\le 1} \chi(b_1(y)px + \cdots + b_{n-1}(y)p^{n-1}x^{n-1} + a_np^nx^n)\, dx, \end{aligned}$$

and b_j is given by (1).

We aim to apply Lemma 6. When $y \neq 0$, we have

$$|b_1(y)p| = |a_1 + 2a_2y + \cdots + na_ny^{n-1}|/p = |ma_m|/p > p.$$

Now if $k > j \ge 2$, then

$$|ma_m| \ge \left|\binom{k-1}{j-1}\right| |ka_k| = \left| j\binom{k}{j} a_k \right|$$

so that

$$|ma_m| \ge \left| ja_j + j\binom{j+1}{j} a_{j+1}y + \cdots + j\binom{n}{j} a_ny^{n-j} \right| = |jb_j(y)|.$$

Hence if $j \ge 2$, then

$$|jb_j(y)p^j| \le \frac{|ma_m|}{p^j} = \frac{|b_1(y)p|}{p^{j-1}} < |b_1(y)p|.$$

Thus by Lemma 6, we have $I_1(y) = 0$ for all $y \neq 0$, so that $I = I_1(0)$. □

Proof of Lemma 1. We use double induction on

$$m = \max\{l : |la_l| \ge |ja_j| \text{ for all } j \neq l\},$$

and

$$r = \max_{1\le j\le n} \log_p |ja_j|.$$

First we note trivially that

$$|I| = \left| \int_{|x|\le 1} \chi(a_1x + \cdots + a_nx^n)\, dx \right| \le \int_{|x|\le 1} |\chi(a_1x + \cdots + a_nx^n)|\, dx = 1.$$

Suppose that $m = 1$. When $r \le 1$,

$$\frac{p^m}{|ma_m|^{1/m}} = \frac{p}{|a_1|} \ge \frac{p}{p} = 1,$$

so we are done. When $r > 1$ we have $|a_1| > p$, and as $|a_1| > |ja_j|$ for all $j > 1$, we obtain the result by Lemma 6 . Now suppose that $m > 1$ and $r \le 2$. Again we are done, as

$$\frac{p^m}{|ma_m|^{1/m}} \ge \frac{p^2}{p^{2/2}} \ge 1.$$

So when $m = 1$ or $r \le 2$, we have the result.

Suppose we have the result when $m \le k-1$ and $r \le s-1$, and suppose that $m = k$ and $r = s$. When $|y| \le 1$, we have

$$\{x : |x| \le 1\} = \{x : |x - y| \le 1\}$$

so that

$$\begin{aligned}|I| &= \left|\int_{|x-y|\le 1} \chi(a_1x+\cdots+a_nx^n)\,dx\right|\\ &= \left|\int_{|h|\le 1} \chi(a_1(h+y)+\cdots+a_n(h+y)^n)dh\right|\\ &= \left|\int_{|h|\le 1} \chi(b_1(y)h+\cdots+b_{n-1}(y)h^{n-1}+a_nh^n)dh\right|,\end{aligned}$$

for all $|y|\le 1$, where b_j is given by (1).

As $m=k$, we have $|ka_k|>|ja_j|$ for all $j>k$. Thus when $|y|\le 1$, we have $|kb_k(y)|>|jb_j(y)|$ for all $j>k$, by Lemma 5. We choose $y=y_1$, so that

$$\max_{1\le j<k}|jb_j(y_1)| = \min_{|y|\le 1}\max_{1\le j<k}|jb_j(y)|.$$

Either $\max_{1\le j<k}|jb_j(y_1)|<|kb_k(y_1)|$ or $\max_{1\le j<k}|jb_j(y_1)|\ge|kb_k(y_1)|$.

When $\max_{1\le j<k}|jb_j(y_1)|<|kb_k(y_1)|$, we have $|kb_k(y_1)|>|jb_j(y_1)|$ for all $j\ne k$, so we can apply Lemma 7 to obtain

$$|I| = \frac{1}{p}\left|\int_{|h|\le 1}\chi(b_1(y_1)ph+\cdots+b_{n-1}(y_1)p^{n-1}h^{n-1}+a_np^nh^n)dh\right|.$$

Now as $\max_{1\le j\le n}|jb_j(y_1)p^j|\le p^{s-1}$, we have

$$r=\max_{1\le j\le n}\log_p|jb_j(y_1)p^j|\le s-1.$$

Since $|kb_k(y_1)|>|jb_j(y_1)|$ for all $j>k$, we have

$$m=\max\{l:|lb_l(y_1)p^l|\ge|jb_j(y_1)p^j| \quad\text{for all}\quad j\ne l\}=k_1\le k.$$

Hence

$$|I|\le\frac{1}{p}\frac{p^{k_1}}{|k_1b_{k_1}(y_1)p^{k_1}|^{1/k_1}}=\frac{p^{k_1}}{|k_1b_{k_1}(y_1)|^{1/k_1}}\le\frac{p^k}{|kb_k(y_1)|^{1/k}},$$

by induction. Finally

$$|I|\le\frac{p^k}{|ka_k|^{1/k}},$$

as $|kb_k(y_1)|=|ka_k|$ by Lemma 5.

When $\max_{1\le j<k}|jb_j(y_1)|\ge|kb_k(y_1)|$ we split the integral into p pieces, so that

$$I=\int_{|x|\le 1}\chi(a_1x+\cdots+a_nx^n)\,dx=\sum_{y=0}^{p-1}\chi(a_1y+\cdots+a_ny^n)I_1(y),$$

where

$$\begin{aligned}I_1(y)&=\int_{|h|\le 1/p}\chi(b_1(y)h+\cdots+b_{n-1}(y)h^{n-1}+a_nh^n)\,dh\\ &=\frac{1}{p}\int_{|x|\le 1}\chi(b_1(y)px+\cdots+b_{n-1}(y)p^{n-1}x^{n-1}+a_np^nx^n)\,dx,\end{aligned}$$

and b_j is given by (1). Now when $|y|\le 1$, we have

$$|kb_k(y)|=|ka_k|>|lb_l(y)|$$

for all $l > k$, by Lemma 5. Hence

$$\max_{1\le j<k} |jb_j(y)| \ge |kb_k(y_1)| = |ka_k| \ge |lb_l(y)|$$

for all $l \ge k$. Thus for $y = 0, \dots, p-1$, there exists $k_1 < k$, where k_1 depends on y, such that

$$|k_1 b_{k_1}(y)| \ge |jb_j(y)|,$$

and

$$|k_1 b_{k_1}(y)p^{k_1}| > |jb_j(y)p^j|$$

for all $j > k_1$. Hence for $y = 0, \dots, p-1$, we have

$$m = \max\{l : |lb_l(y)p^l| \ge |jb_j(y)p^j| \text{ for all } j \ne l\} = k_1 < k.$$

Thus

$$|I_1(y)| \le \frac{1}{p}\frac{p^{k_1}}{|k_1 b_{k_1}(y)p^{k_1}|^{1/k_1}} = \frac{p^{k_1}}{|k_1 b_{k_1}(y)|^{1/k_1}},$$

by induction. Now as

$$\frac{p^{k_1}}{|k_1 b_{k_1}(y)|^{1/k_1}} \le \frac{p^{k-1}}{|kb_k(y)|^{1/k}} = \frac{p^{k-1}}{|ka_k|^{1/k}},$$

by Lemma 5, we have

$$|I| \le \sum_{y=0}^{p-1} |I_1(y)| \le p\frac{p^{k-1}}{|ka_k|^{1/k}} = \frac{p^k}{|ka_k|^{1/k}},$$

and we are done. □

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